Average light velocities in periodic media

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Electromagnetic Bloch modes are used to describe the field distribution of light in periodic media that cannot be adequately approximated by effective macroscopic media. These modes explicitly take into account the spatial modulation of the medium and therefore contain the full physical information at any specific location in the medium. For instance, the propagation velocity of light can be determined locally, and it is *not* an invariant of space, as it is often implicitly assumed when definitions such as that of the group velocity $v_{\mathrm{gr}} = d\omega/dk$ are used (where ω is the angular frequency and k is the Bloch index of a monochromatic mode). Spatially invariant light velocities can only be expected if the medium is assumed to show an effective behavior similar to a homogeneous material (where a plane-wave ansatz would be more appropriate). This inevitably leads to the question: what exactly is $d\omega/dk$ of a Bloch mode, if it is not the group velocity? The answer is the *average* group velocity. This is not a trivial observation, and it has to be taken into account, for instance, when the enhancement of nonlinear effects induced by slow light is estimated. The example of a Kerr nonlinearity is studied, and we show formally that using the average group velocity can lead to an underestimation of the effect. Furthermore, this article critically reviews the concepts of energy and phase velocity. In particular, the different interpretations of phase velocity that exist in the literature are unified using a generic definition of the quantity.

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1. INTRODUCTION

The concepts of group, energy, and phase velocity in periodic systems were first discussed in detail in the pioneering works of Yariv and Yeh [1] and Yeh [2]. For the group and energy velocity, Yariv and Yeh showed that their definitions are equivalent in systems composed of nonabsorbing materials. This result led to the general acceptance of their definitions. The phase velocity, on the other hand, was discussed more controversially. In the photonic crystal community, there appears to be a general consensus that phase velocity has no definite meaning in the context of Bloch modes [3,4]. A careful review of the three velocities reveals that there are a few inconsistencies in their definitions. It is the purpose of this article to clarify them. Notably, the following issues will be addressed:

(1) Bloch ansatz versus effective macroscopic behavior: choosing the Bloch representation means that we expect a behavior different from that in an effective homogeneous material (otherwise we would make a plane-wave ansatz using an effective permittivity). *A priori*, we expect the physical properties of our mode solutions to be dependent on the spatial coordinate along the direction of periodicity. If we intend to use definitions such as the standard definition for the group velocity, $v_{\mathrm{gr}} = (dk/d\omega)^{-1}$, we have to explain why they are invariant in space.

(2) Bloch index k versus plane-wave propagation constant k: a Bloch mode of frequency ω is a plane wave, $e^{i[k(\omega)x-\omega t]}$, superimposed by a periodic complex function $u_\omega(x)$. The notation will be introduced in more detail below. The complex function $u_\omega(x)$ carries a space-dependent phase component. Hence, in contrast to plane-wave modes, the value of k does not contain the full phase information. There is no one-to-one analogy between wave number and Bloch index [5]. Using the dispersion relation $k(\omega)$ of a Bloch mode for the definition of any physical quantity means that the phase information hidden in $u_\omega(x)$ is neglected. We will show that, for the group velocity, neglecting this information is equivalent to taking an average over a unit cell of the periodic medium. Regarding the phase velocity, neglecting phase information inevitably leads to definitions whose physical significance and relevance are questionable.

The article is organized as follows: first, the notation of the Bloch modes used throughout the article is introduced in Section 2. Then the definitions of group, phase, and energy velocity will be discussed in subsequent sections. For the definition of group velocity in Section 3, a *group* or *pulse* will be formed using an arbitrary spectrum of Bloch modes, and its evolution will be studied. Unlike the standard derivations, this approach truly justifies the term *group velocity* of the resulting definition because it does not rely on the validity of analogies between the Bloch index and the wave number of a plane wave. We will obtain a quantity that fluctuates along the propagation direction and whose average value over a unit cell will conform to the standard definition, $(dk/d\omega)^{-1}$. Since there are many ways to compute an average value, we will verify that the obtained average is physically meaningful. We will also point out that ignoring the spatial fluctuations of the group velocity can result in underestimations of nonlinear effects in slow-light media. For the phase velocity, we will introduce a straightforward definition in Section 4 that takes into account the full phase information contained in a Bloch mode. We will show that the average of the resulting quantity over a unit cell is in accordance with the definitions given by Yariv and Yeh using their so-called *principal value of* k [1,2]. For completeness sake, the energy velocity is discussed in Section 5. The standard definition is $v_{\mathrm{en}} = \langle\langle S\rangle_T\rangle_\Gamma / \langle\langle \mathcal{U}\rangle_T\rangle_\Gamma$, where S is the component in propagation direction of the Poynting vector $\mathbf{S}$, $\mathcal{U}$ is the energy density, and $\langle\cdot\rangle_T$ and $\langle\cdot\rangle_\Gamma$ denote averages over periods in time and space, respectively. Since this article puts a lot of emphasis on physically meaningful averages, we briefly want to address the seemingly unphysical procedures of averaging an energy density over time and averaging an energy flux over space along the propagation direction. Finally, all definitions are summarized and discussed in Section 6, and conclusive remarks are made in Section 7.

2. NOTATION

We will distinguish between periodic and homogeneous dielectric systems, according to their symmetry properties. For systems with a continuous translational symmetry along one particular direction, extensive literature is available [7,8]. We will refer to this type of system as *homogeneous along one direction* or simply *homogeneous*. Monochromatic electromagnetic waves propagating along this direction of translational symmetry (here the x axis) are characterized by a harmonic evolution, i.e., the fields (we will use the $\mathbf{H}$ field) can be written as

$$\mathbf{H}_\omega = \mathbf{h}_\omega(y,z)e^{i[k(\omega)x-\omega t]}, \tag{1}$$

where ω is the angular frequency, $k(\omega)$ is the wave number, and $\mathbf{h}_\omega(y,z)$ represents the field distribution in the transverse plane.

In the present article, the focus shall be on systems with a *periodic* modulation of the permittivity along the propagation direction of the electromagnetic waves. In other words, the continuous translational symmetry of homogeneous media is replaced by a discrete translational symmetry. In particular, we shall be concerned with periods Γ on the order of the wavelength λ of the propagating waves. This means that no effective macroscopic behavior can be assumed. For the present purposes, it is sufficient to consider systems composed of isotropic, nonabsorbing dielectric materials with negligible material dispersion. For the permittivity, we have $\varepsilon(x+\Gamma,y,z) = \varepsilon(x,y,z)$, where Γ is the period of the modulation. Monochromatic waves propagating along the direction of periodicity are referred to as Bloch modes. They can be written as

$$\mathbf{H}_\omega = \mathbf{h}_\omega(y,z)u_\omega(x)e^{i[k(\omega)x-\omega t]}, \tag{2}$$

where we call $k(\omega)$ the Bloch index, and $u_\omega(x)$ is a periodic complex function with period Γ.

3. GROUP VELOCITY

A. Definition

There are several ways to motivate a definition of group velocity. A well-founded motivation must involve a group or packet of waves that are carried by the eigenmode(s) of the studied system according to the spectral content of the excitation signal [9]. The simplest example [10] of a packet is the beating of only two modes. In a homogeneous system, the two modes might be written as $A\cos(k_1x-\omega_1t)$ and $A\cos(k_2x-\omega_2t)$, with $k_1 = k+\Delta k$, $k_2 = k-\Delta k$, $\omega_1 = \omega+\Delta\omega$, $\omega_2 = \omega-\Delta\omega$, and amplitude A. The sum of the two waves is $2A\cos(kx-\omega t)\cos(\Delta kx-\Delta\omega t)$, and we can easily distinguish between the velocity of the phase fronts $v_{\mathrm{ph}} = \omega/k$ and the envelope $v_{\mathrm{gr}} = \Delta\omega/\Delta k \to d\omega/dk$ for $\Delta k \to 0$. Another typical example of a packet, which is often used to motivate the definition of group velocity, is the Gaussian pulse, formed by a continuous spectrum of harmonic waves. Again, an analytical treatment reveals that the envelope propagates at the group velocity $v_{\mathrm{gr}} = d\omega/dk$, and it retains a Gaussian shape if third and higher order dispersion terms can be neglected [11]. Generally speaking, the concept of group velocity is only applicable to narrow-band signals.

Since Bloch modes are harmonic in time, the concept of pulse formation in the time domain is analogous to homogeneous systems. At a fixed location in space, the temporal evolution of a pulse can be constructed from a continuous spectrum of monochromatic modes by Fourier theory. Therefore, let's consider an arbitrary group of Bloch modes whose spectrum shall be given by the complex function $G(\omega)$, which contains the amplitude of all components as well as their phase relation at an initial position $x=0$. The temporal evolution of the field at $x=0$ is described by (dependencies on y and z coordinates are omitted for simplicity):

$$\mathbf{H}(0,t) = \int G(\omega)\mathbf{h}_\omega u_\omega(0)e^{-i\omega t}\mathrm{d}\omega. \tag{3}$$

After letting the group propagate for a fixed time $\bar{t}$, we get the following temporal evolution at point $\bar{x}$:

$$\mathbf{H}(\bar{x},t+\bar{t}) = \int G(\omega)\mathbf{h}_\omega u_\omega(\bar{x})e^{i[k(\omega)\bar{x}-\omega(t+\bar{t})]}\mathrm{d}\omega. \tag{4}$$

In a system with little dispersion, the temporal evolution at point $\bar{x}$ will be similar to that at $x=0$, up to a time shift $\bar{t}$ and possibly a phase shift. Assuming an undistorted propagation of the group, this can be stated as

$$\mathbf{H}(\bar{x},t+\bar{t}) = \mathbf{H}(0,t)e^{i\varphi_0} \tag{5}$$

for all t and some constant phase shift φ_0. If $\bar{x}$ is an integer multiple of Γ, then inserting Eqs. (3) and (4) in Eq. (5) yields the requirement

$$k(\omega)\bar{x} - \omega\bar{t} = \varphi_0 \tag{6}$$

for all ω. If we define $\phi(\omega,\bar{x},\bar{t}) = k(\omega)\bar{x}-\omega\bar{t}$, then Eq. (6) can be rewritten as

$$\frac{d^n\phi}{d\omega^n} = 0 \quad \text{for all } n \geq 1. \tag{7}$$

Equation (7) with $n=1$ defines the trajectory $\bar{x}(\bar{t})$ of the group, whereas the equations with $n>1$ can only be fulfilled if the material properties of the system allow for an undistorted pulse propagation. If the material requirements are not fulfilled exactly, we can nevertheless proceed to use $d\phi/d\omega = 0$ to define an *approximate* trajectory of the pulse and the corresponding group velocity, keeping in mind that the group will be distorted. This definition of the group velocity will be reasonable over a limited range of $\bar{x}$. The size of this range depends on the material properties and the initial pulse shape. The condition $d\phi/d\omega = 0$ is equivalent to

$$\frac{dk}{d\omega}(\omega)\bar{x} - \bar{t} = 0. \tag{8}$$

For an undistorted group, Eq. (8) is valid for all $\bar{x} = m\Gamma$, $m \in \mathbb{N}$. Therefore, we can define the average group velocity

$$\langle v_{\mathrm{gr}} \rangle = \frac{\bar{x}}{\bar{t}} = \left(\frac{dk}{d\omega}(\omega)\right)^{-1}. \tag{9}$$

This confirms that the commonly used definition is reasonable, since in most experiments it is the average velocity over macroscopic distances that is of interest. However, at this point we can make no statement about potential fluctuations of the group velocity between the periods.

B. Between the Periods

So far, we have only considered locations that are integer multiples of the spatial period. To gain further insight about what happens *between* the periods, we need some knowledge about the periodic function $u_\omega(x)$. First, we represent $u_\omega(x)$ in polar form, $u_\omega(x) = |u_\omega(x)|e^{i\varphi(\omega,x)}$, where $\varphi(\omega, x)$ is a real function satisfying the relation $\varphi(\omega, x + \Gamma) - \varphi(\omega, x) = 2\pi m$, $m \in \mathbb{N}$, for all x. This lets us rewrite the Bloch mode of Eq. (2) as

$$\begin{aligned}\mathbf{H}_\omega &= \mathbf{h}_\omega(y,z)|u_\omega(x)|e^{i[k(\omega)x+\varphi(\omega,x)-\omega t]}\\ &= \mathbf{h}_\omega(y,z)|u_\omega(x)|e^{i\Phi_\omega(x,t)}\end{aligned} \tag{10}$$

with $\Phi_\omega(x,t) = k(\omega)x + \varphi(\omega,x) - \omega t$. Now we consider the *artificial* example of a system in which $|u_\omega(x)|$ is a constant with respect to x, i.e., $u_\omega(x) = u_\omega e^{i\varphi(\omega,x)}$. Assuming this form of $u_\omega(x)$ is instructive because it allows us to use the method of stationary phase [12] to define the trajectory of the group in a similar way as above. Equations (3) and (4) rewrite as

$$\mathbf{H}(0,t) = \int G(\omega)\mathbf{h}_\omega u_\omega e^{i\Phi_\omega(0,t)}\mathrm{d}\omega \tag{11}$$

and

$$\mathbf{H}(\bar{x}, t+\bar{t}) = \int G(\omega)\mathbf{h}_\omega u_\omega e^{i\Phi_\omega(\bar{x},t+\bar{t})}\mathrm{d}\omega. \tag{12}$$

With $\phi(\omega,\bar{x},\bar{t}) = \Phi_\omega(\bar{x}, t+\bar{t}) - \Phi_\omega(0,t)$ and $\varphi(\omega,0) = 0$ (w.l.o.g.), the stationary phase requirement, $d\phi/d\omega = 0$, translates into

$$\frac{dk}{d\omega}(\omega)\bar{x} + \frac{d\varphi}{d\omega}(\omega,\bar{x}) - \bar{t} = 0. \tag{13}$$

By taking a time-derivative, the group velocity $v_{\mathrm{gr}} = d\bar{x}/d\bar{t}$ can be extracted,

$$\frac{d}{d\bar{x}}\left(\frac{dk}{d\omega}(\omega)\bar{x} + \frac{d\varphi}{d\omega}(\omega,\bar{x})\right)\cdot v_{\mathrm{gr}} = 1. \tag{14}$$

The group velocity, as defined in Eq. (14), depends on the location in space. We can compute the average group velocity along the trajectory,

$$\begin{aligned}\langle v_{\mathrm{gr}}(\omega)\rangle^{-1} &= \frac{1}{\Gamma}\int_x^{x+\Gamma} v_{\mathrm{gr}}(\omega,\bar{x})^{-1}\mathrm{d}\bar{x}\\ &= \frac{1}{\Gamma}\int_x^{x+\Gamma}\frac{d}{d\bar{x}}\left(\frac{dk}{d\omega}(\omega)\bar{x} + \frac{d\varphi}{d\omega}(\omega,\bar{x})\right)\mathrm{d}\bar{x}\\ &= \frac{dk}{d\omega}(\omega) + \frac{1}{\Gamma}\frac{d}{d\omega}(\varphi(\omega,x+\Gamma) - \varphi(\omega,x))\\ &= \frac{dk}{d\omega}(\omega).\end{aligned} \tag{15}$$

Using this spatial average, we again reproduce a result that is formally identical to the standard definition [Eq. (9)]. However, our example shows that, if the phase of $u_\omega(x)$ is dependent on x, then the group velocity fluctuates between the periods. From a conceptual point of view, this is an important insight. But also experimentally, the fluctuations in v_{gr} along x can have an impact. For instance, when the enhancement of nonlinear effects through slow light is to be estimated, the average group velocity might not necessarily be a precise reference. Let's briefly consider the case of nonlinear interaction in a Kerr medium. According to Krauss [13], the refractive index change Δn induced by an incident light pulse is proportional to $(1/v_{\mathrm{gr}})^2$. Using a proportionality constant B (containing the nonlinear coefficient of the Kerr medium), we can write $\Delta n = B(1/v_{\mathrm{gr}})^2$, and the phase change accumulated over a period, Γ, is $\Delta\varphi = \Gamma\frac{\omega}{c}B(1/v_{\mathrm{gr}})^2$. Now if we let v_{gr} be a function of the x coordinate, we get

$$\begin{aligned}\Delta\varphi &= \int_0^\Gamma \frac{\omega}{c}B(1/v_{\mathrm{gr}}(x))^2\mathrm{d}x = \Gamma\frac{\omega}{c}B\langle(1/v_{\mathrm{gr}})^2\rangle\\ &\geq \Gamma\frac{\omega}{c}B\langle 1/v_{\mathrm{gr}}\rangle^2\\ &\geq \Gamma\frac{\omega}{c}B(1/\langle v_{\mathrm{gr}}\rangle)^2.\end{aligned} \tag{16}$$

The last expression in Eq. (16) is the result to be expected in the absence of local variations in v_{gr}. It is inferior to the result obtained for a fluctuating v_{gr}. This suggests that the use of an averaged group velocity can lead to an underestimation of the Kerr effect. A similar conclusion can be drawn by considering the intensity distribution of a Bloch mode in a periodic medium. There is a modulation in intensity according to the medium's geometry. The effect of local intensity maxima and minima on the accumulated Kerr phase can be computed in a manner analogous to Eq. (16):

$$\Delta\varphi = \int_0^\Gamma \frac{\omega}{c}\tilde{B}I(x)^2\mathrm{d}x, \tag{17}$$

where $I(x)$ is the field intensity and $\tilde{B}$ is the corresponding proportionality constant that contains the nonlinearity coefficient. To estimate the order of magnitude of the effect, one can assume that $I(x) = I_0 - \varepsilon$ for $x \in [0, \Gamma/2)$ and $I(x) = I_0 + \varepsilon$ for $x \in [\Gamma/2, \Gamma)$, where ε represents the fluctuation of $I(x)$ around its average value I_0. This yields an accumulated phase of $\Delta\varphi = \Gamma\frac{\omega}{c}\tilde{B}(I_0^2 + \varepsilon^2)$, compared to $\Delta\varphi = \Gamma\frac{\omega}{c}\tilde{B}I_0^2$, which would be obtained for a constant intensity of I_0. Hence, using a constant intensity leads to a relative underestimation of the phase $\Delta\varphi$ by a factor of $(\varepsilon/I_0)^2$. This behavior depends on the nature of the nonlinear effect under consideration (e.g., assuming a third-order nonlinear effect in this simplified treatment would result in a dominant term that is linear in ε/I_0).

Future numerical and experimental investigations will have to show how significant the effect of local variations in v_{gr} is in specific experimental situations. However, even in cases where the effect is substantial, it might be challenging to measure it experimentally. Other important effects such as optical losses, dispersion, and mode shape variations for different values of $\langle v_{gr}\rangle$ can make it difficult to quantify each contribution individually. For instance, in the context of four-wave mixing, it took quite a lot of effort to experimentally verify the trend that the conversion efficiency increases with the fourth power of $1/v_{gr}$ [14].

C. Remark About Averaging of Physical Quantities

In Eq. (15), a spatial average is taken of the quantity v_{gr}^{-1}. Why not take the average of v_{gr} directly? The answer is simple, but worth recapitulating. Physically meaningful values are obtained by either averaging v_{gr} over time or v_{gr}^{-1} over space. This can be seen by considering a 1D trajectory $r(t)$ of a particle. The average velocity $\langle v\rangle$ of the particle traveling a distance Δr during the time interval Δt is given by

$$\langle v\rangle = \frac{\Delta r}{\Delta t} = \frac{1}{\Delta t}\int 1\mathrm{d}r = \frac{1}{\Delta t}\int \frac{dr}{dt}\mathrm{d}t = \frac{1}{\Delta t}\int v\mathrm{d}t. \quad (18)$$

The integration of v has to be performed over time. If, instead, we compute the average over distance, we get

$$\frac{1}{\Delta r}\int v\mathrm{d}r = \frac{1}{\Delta r}\int \frac{dr}{dt}\mathrm{d}r = \frac{1}{\Delta r}\int \frac{dr}{dt}\frac{dr}{dt}\mathrm{d}t = \frac{\Delta t}{\Delta r}\left(\frac{1}{\Delta t}\int v^2\mathrm{d}t\right) = \frac{\langle v^2\rangle}{\langle v\rangle} \geq \langle v\rangle. \quad (19)$$

Hence the average of v over distance results in an overestimated velocity. On the other hand, if we average v^{-1} over distance, we get

$$\frac{1}{\Delta r}\int v^{-1}\mathrm{d}r = \frac{1}{\Delta r}\int \frac{dt}{dr}\mathrm{d}r = \frac{1}{\Delta r}\int 1\mathrm{d}t = \frac{\Delta t}{\Delta r} = \langle v\rangle^{-1}, \quad (20)$$

which is the physically meaningful result. We will meet the issue of averaging again in the context of energy velocity in Section 5.

4. PHASE VELOCITY

From the polar representation [Eq. (10)] of a Bloch mode, we can formally define a phase velocity, i.e., the velocity at which the phase fronts [15] (lines of constant phase) propagate in space [16]:

$$v_{ph}(x) = -\frac{\partial_t\Phi_\omega}{\partial_x\Phi_\omega} = \frac{\omega}{k(\omega) + \partial_x\varphi(\omega, x)}. \quad (21)$$

The phase velocity is a function that is periodic along x. Corresponding to an interpretation often found in the literature, it can be decomposed into Fourier components such that a phase velocity can be attributed to each Bloch component of the mode.

Following the same averaging procedure as in Eq. (15), we obtain the average phase velocity

$$\langle v_{ph}\rangle = \frac{\omega}{k(\omega) + m\frac{2\pi}{\Gamma}}, \quad (22)$$

where m is an integer number as defined in Section 3.B. The value $k + m(2\pi/\Gamma)$ corresponds to what Yariv and Yeh called the *principal value of k* in their definition of the phase velocity [1,2,17]. The corresponding mth Fourier component of the Bloch mode was referred to as the *fundamental space harmonic*. The average phase velocity $\langle v_{ph}\rangle$ as defined in Eq. (22) is equal to the phase velocity of this fundamental component. Only in the long-wavelength regime, where the system behaves like an effective homogeneous material, the fundamental component is dominant and by itself accurately approximates the Bloch mode.

These considerations unify the different interpretations of phase velocity that exist in the context of Bloch modes using a single unambiguous definition. Again, the essential new insight is that the standard definition corresponds to an *averaged* quantity.

The phase velocity is mainly of conceptual interest and is hardly accessible experimentally. While for a homogeneous system, it can be linked to the refractive index of the material, this relation is not readily established for a periodic system. The phenomenon of wave refraction at interfaces between a homogeneous medium and a photonic crystal is discussed in detail in [18]. Depending on the angle of incidence, the wave can be split up into more than one component (similar to birefringence), or it can be deflected in unusual ways (e.g., negative refraction) through a mechanism that is more closely related to diffraction than to refraction (the zeroth diffraction order is suppressed by a photonic band gap).

5. ENERGY VELOCITY

The objective of this section is to produce a definition of the energy velocity without having to perform averages of quantities that are not physically motivated, such as the average of energy density over time or the average of energy flux along the propagation direction. To our knowledge, there is no similar treatment reported in the literature, although it is essential for a full understanding of the physical meaning behind the definition of energy velocity.

Consider a wave propagating along the x coordinate of a nonabsorbing system. The system shall be either homogeneous or periodic along x. Further consider a box of length L along the x axis and with a cross section A as shown in Fig. 1. We assume a 2D mode confinement in the y and z plane and that A is large compared to the transverse mode size, such that energy fluxes through surfaces perpendicular to A can be neglected. If the energy contained in the volume $V = L \cdot A$ flows out through the area A within time τ, then we can define the energy velocity as $v_{en} = L/\tau$. This definition is on solid ground if τ can be unambiguously determined, i.e., if the energy in V is constant in time. We will see that for harmonic waves, this can be guaranteed by choosing the length of the box to be an integer multiple of the wavelength. For Bloch modes, this is not possible, and the issue has to be solved by choosing large values of L. Since an integration over the volume V is involved, v_{en} cannot be understood as a local quantity. We will think of it as an *average* energy velocity and denote it $\langle v_{\mathrm{en}}\rangle$.

Let the energy density be denoted by $\mathcal{U}(x,y,z,t)$ and the energy flux density in propagation direction by $S(x,y,z,t)$. The total energy flowing through the area A during a time interval τ is $E(\tau)=\int_A\int_0^\tau S\mathrm{d}t\mathrm{d}y\mathrm{d}z$, and the total energy contained in the box is $E_{\mathrm{box}}=\int_V\mathcal{U}\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where $V=L\cdot A$. In the case of a harmonic time dependence of the fields with period T, we can write $\int_0^\tau S\mathrm{d}t=\tau\langle S\rangle_T$ for large τ. Moreover, if $\mathcal{U}$ is the product of periodic functions along x, we have $\int_V\mathcal{U}\mathrm{d}x\mathrm{d}y\mathrm{d}z=L\cdot\int_A\langle\mathcal{U}\rangle_L\mathrm{d}y\mathrm{d}z$ for large L. The energy velocity can now be defined as $\langle v_{\mathrm{en}}\rangle=L/\tau$, for the time τ satisfying $E(\tau)=E_{\mathrm{box}}$, which is equivalent to $\tau\int_A\langle S\rangle_T\mathrm{d}y\mathrm{d}z=L\int_A\langle\mathcal{U}\rangle_L\mathrm{d}y\mathrm{d}z$. This justifies the definition

$$\langle v_{\mathrm{en}}\rangle=\frac{\int_A\langle S\rangle_T\mathrm{d}y\mathrm{d}z}{\int_A\langle\mathcal{U}\rangle_L\mathrm{d}y\mathrm{d}z}. \tag{23}$$

The time τ can only be unambiguously determined if E_{box} (and therefore $\langle\mathcal{U}\rangle_L$) is constant in time. This is the case, e.g., for a plane-wave field with $\mathcal{U}\sim\cos^2(kx-\omega t)$, if we choose the size of the box such that L is an integer multiple of $2\pi/k$.

The standard textbook definition [10,11,19] of energy velocity in homogeneous media is $\langle v_{\mathrm{en}}\rangle=\langle S\rangle_T/\langle\mathcal{U}\rangle_T$. Integrals over the surface A might be added if the mode is confined. The difference to our definition is the time average of $\mathcal{U}$ instead of a space average. For harmonic waves with $\mathcal{U}\sim\cos^2(kx-\omega t)$, the two definitions are equivalent since both averages return the same result. To our understanding, the time average of $\mathcal{U}$ in the textbook definition can only be justified through this equivalence.

For a Bloch mode, it can be shown that $\lim_{L\to\infty}\langle\mathcal{U}\rangle_L=\langle\langle\mathcal{U}\rangle_T\rangle_\Gamma$. This allows us to rewrite Eq. (23) as

$$\langle v_{\mathrm{en}}\rangle=\frac{\int_A\langle S\rangle_T\mathrm{d}y\mathrm{d}z}{\int_A\langle\langle\mathcal{U}\rangle_T\rangle_\Gamma\mathrm{d}y\mathrm{d}z}. \tag{24}$$

The denominator in Eq. (24) is constant along the x coordinate, but what about the numerator? The time-averaged energy-flux density $\langle S\rangle_T$ certainly has to be considered dependent on x. On the other hand, the integral over A has to be independent of x because otherwise the energy conservation would be violated (we could easily conceive a box similar to the one in Fig. 1 with a constant energy outflow through the surface A at x_0 that exceeds the energy inflow through an equivalent surface at x_0-L). Therefore adding a spatial

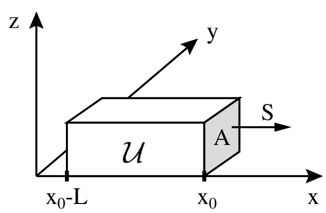

Fig. 1. Box of length L and cross-section area A. An electromagnetic wave propagates along x. The energy density in the box is $\mathcal{U}$, and the energy flux density through the area A is $S=\mathbf{S}\cdot\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is the unit vector along the x coordinate.

average $\langle\cdot\rangle_\Gamma$ around $\langle S\rangle_T$ in the numerator of Eq. (24) makes no difference, and we arrive at an expression similar to the standard definition given by Yariv and Yeh.

To conclude this section, we emphasize again that the standard definitions of group and energy velocity have to be understood as averaged quantities. As pointed out in Section 3.C, it is essential that the averaging procedure is motivated by physical arguments. In this sense, the findings of Sections 3 and 5 retrospectively form an indispensable basis for considerations such as the proof of the equivalence $v_{\mathrm{gr}}=v_{\mathrm{en}}$ in lossless systems [1,2] or the decomposition of v_{en} into weighted contributions from each Fourier component of a Bloch mode [20]. In fact, the equivalence result, $v_{\mathrm{gr}}=v_{\mathrm{en}}$, only now unfolds its full physical meaning.

6. DISCUSSION

Table 1 summarizes the results of Sections 3–5. If we motivate the definitions of all velocities from first principles, we can obtain a fully consistent set. If we do not explicitly consider the long-wavelength regime, we have to allow the velocities to vary along the direction of propagation. For the group and phase velocity, we derived the x-dependent expressions. For the group velocity, we assumed a special form of the periodic function $u_\omega(x)=u_\omega e^{i\varphi(\omega,x)}$. If $|u_\omega(x)|$ contains an additional x dependence, then the pulse shape is distorted between the periods, and defining a local group velocity is no longer straightforward. However, the average group velocity can still be defined, and it corresponds to the standard definition found in the literature. In fact, both the average group velocity and the average phase velocity derived here agree with the definitions given by Yariv and Yeh [1,2], except that they have not previously been recognized as being averaged quantities. We can conclude that, if the Bloch index k is treated as a full analogue to the wave number of a plane wave, then the fact that k does not contain the full phase information of the mode naturally leads to averaged quantities. This result is nontrivial, and we believe it has not received due attention, possibly because of its intuitive appeal.

The averaged velocities often provide an appropriate description of the physics of a waveguide, and local phase fluctuations between the periods can often be neglected. However, there can be situations where the microscopic phase properties will matter and where we must be alert to subtle effects of the periodicity. For instance, problems can arise in measurements involving interference between two Bloch modes. In particular, we believe that for the experimental measurement of the group velocity using interference fringe patterns, micro-

Table 1. Definitions of the Group, Energy, and Phase Velocity of Bloch Modes, According to Sections 3–5[a]

	x Dependent	Average
v_{ph}	$\frac{\omega}{k(\omega)+\partial_x\varphi(\omega,x)}$	$\frac{\omega}{k(\omega)+m\frac{2\pi}{\Gamma}}$
v_{gr}	$\left[\frac{d}{dx}\left(\frac{dk}{d\omega}(\omega)x+\frac{d\varphi}{d\omega}(\omega,x)\right)\right]^{-1}$	$\left(\frac{dk}{d\omega}(\omega)\right)^{-1}$
v_{en}		$\frac{\int_A\langle S\rangle_T\mathrm{d}y\mathrm{d}z}{\int_A\langle\langle\mathcal{U}\rangle_T\rangle_\Gamma\mathrm{d}y\mathrm{d}z}$

[a]For the x dependent group velocity, the special form $u_\omega(x)=u_\omega e^{i\varphi(\omega,x)}$ is assumed.

scopic effects should be taken into account. However, an analytical treatment of such interference effects is rather challenging because interference phenomena typically involve structures with nonperfect periodicity (e.g., a structure terminated at one end), and a treatment in terms of pure Bloch modes is, therefore, hardly appropriate. Another example where microscopic propagation properties can be of importance is the estimation of nonlinear effects in slow-light waveguides, as addressed in Section 3.B. We have seen that local fluctuations of the group velocity can affect the strength of a Kerr nonlinearity. In cases like this, the field patterns of the excited Bloch modes have to be investigated to guarantee that local intensity maxima are accurately taken into account.

7. CONCLUSION

We have seen that the standard definitions of group and energy velocity of Bloch modes in periodic systems represent meaningful quantities in the sense of average values. Both definitions were motivated from first principles, and the averaging procedures were critically reviewed for the first time. The well-known equivalence result, $v_{\mathrm{gr}} = v_{\mathrm{en}}$, now displays its full physical meaning.

The example of nonlinear effects in a Kerr medium suggests there is a practical relevance of our findings. We demonstrated formally that using the average group velocity $\langle v_{\mathrm{gr}} \rangle$ can lead to an underestimation of the nonlinear effect.

Using a polar representation of the Bloch mode and taking the phase evolution explicitly into account, the phase velocity can be defined in a straightforward manner. The definition is fully consistent with that of the group velocity in the sense that the average values of both quantities correspond to the definitions that result if the Bloch index k is treated as a full analogue to the wave number of a plane wave. As it turns out, the fact that the Bloch index does not contain the full phase information of the mode naturally leads to averaged quantities.

REFERENCES AND NOTES

1. A. Yariv and P. Yeh, “Electromagnetic propagation in periodic stratified media. II. Birefringence, phase matching, and x-ray lasers,” J. Opt. Soc. Am. **67**, 438–448 (1977).
2. P. Yeh, “Electromagnetic propagation in birefringent layered media,” J. Opt. Soc. Am. **69**, 742–756 (1979).
3. K. Sakoda, *Optical Properties of Photonic Crystals*, 2nd ed., Springer Series in Optical Sciences (Springer, Berlin, 2005).
4. J. D. Joannopoulos and S. G. Johnson, *Photonic Crystals*, 2nd ed. (Princeton University, 2008).
5. A simple way of illustrating the difference in nature between wave vector and Bloch index is to consider the analogy to solid-state physics. The momentum of a given electronic Bloch state $|\psi\rangle$ is not simply given by $\hbar$ multiplied by the Bloch index of the mode (as is the case for the plane-wave momentum of a free electron), but by $\langle\psi|\hat{\mathbf{p}}|\psi\rangle$, which contains a weighted sum over all plane-wave components [6].
6. N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Brooks/Cole, 1976).
7. K. E. Oughstun, *Electromagnetic and Optical Pulse Propagation* 1: *Spectral Representations in Temporally Dispersive Media*, Springer Series in Optical Sciences (Springer, 2006).
8. K. E. Oughstun, *Electromagnetic and Optical Pulse Propagation* 2: *Temporal Pulse Dynamics in Dispersive*, *Attenuative Media*, Springer Series in Optical Sciences (Springer, 2009).
9. Sometimes an individual Bloch mode is mistakenly interpreted as a pulse train formed by a harmonic wave and a periodic envelope function $u_\omega(x)$. This interpretation cannot reflect the phenomenon of pulse propagation, since $u_\omega(x)$ is stationary in space (it is independent of time).
10. L. Brillouin, *Wave Propagation and Group Velocity*, Pure and Applied Physics (Academic, 1960).
11. J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999).
12. M. J. Lighthill, “Group velocity,” IMA J. Appl. Math. **1**, 1–28 (1965).
13. T. F. Krauss, “Slow light in photonic crystal waveguides,” J. Phys. D **40**, 2666–2670 (2007).
14. J. Li, L. O’Faolain, I. H. Rey, and T. F. Krauss, “Four-wave mixing in photonic crystal waveguides: slow light enhancement and limitations,” Opt. Express **19**, 4458–4463 (2011).
15. It is sometimes argued that phase fronts cannot be unambiguously defined because a Bloch mode is the result of multiple plane waves (some of them counterpropagating). Equation (10) clearly shows that this implication is incorrect.
16. G. B. Whitham, “Group velocity and energy propagation for three-dimensional waves,” Commun. Pure Appl. Math. **14**, 675–691 (1961).
17. Note that, due to the periodicity of $u_\omega(x)$ in Eq. (2), replacing k with $k + m(2\pi/\Gamma)$ will result in a new function $u_\omega(x)e^{-im(2\pi/\Gamma)x}$ with a fully periodic phase function $\varphi(\omega, x)$.
18. M. Notomi, “Theory of light propagation in strongly modulated photonic crystals: refractionlike behavior in the vicinity of the photonic band gap,” Phys. Rev. B **62**, 10696–10705 (2000).
19. L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed., Course of Theoretical Physics (Pergamon, 1984).
20. B. Lombardet, L. A. Dunbar, R. Ferrini, and R. Houdré, “Fourier analysis of Bloch wave propagation in photonic crystals,” J. Opt. Soc. Am. B **22**, 1179–1190 (2005).