Discharge time for a capacitor

Joseph R. Cleveland and Stanley Hirschi
Department of Physics, Central Michigan University, Mt. Pleasant, Michigan 48859
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The complete discharge of a capacitor through a real resistor is limited by thermal noise. A meaningful discharge time $t_d$ is defined by equating the charge across the capacitor in a noiseless RC circuit to the rms charge fluctuation in a noisy circuit $Q_0(t_d) = \langle (\delta Q)^2_0 \rangle^{1/2}$. Three derivations are reviewed for the well-known result $\langle (\delta Q)^2 \rangle = kTC$. One derivation shows explicitly that $\delta Q(t)$ can be determined with the noise current $i_n(t)$ as its source. Several examples of potential interest in the classroom are discussed.

I. INTRODUCTION

In most introductory textbooks the discharge of an initially charged capacitor $C$ through a resistance $R$ is discussed in terms of a characteristic decay time $\tau = RC$, for which the charge has decayed to 37% $(1/e)$ of its initial value $Q_0$. It has been our experience that a few puzzled students invariably ask, “But when is the capacitor completely discharged?” This question remains unanswered in the usual textbook material, because the exponential decay of a noiseless RC circuit approaches zero only as time approaches infinity. In this article we present self-consistent arguments that can form a basis for responding to these students. In particular, we show how thermal noise in the resistor defines a natural (and finite) end to the decay process. While the detailed discussion and analysis is beyond that suitable for the introductory course, the results have been presented successfully to both beginning and advanced undergraduates.

For charge decay through any real resistor, complete discharge is masked by the thermal noise voltage $\delta V(t)$ produced by random thermal motions of charge carriers in the resistor. The effect of the noise voltage is to induce charge fluctuations $\delta Q(t)$ on the capacitor. Thus, the charge on the capacitor is described as

$$Q(t) = Q_0(t) + \delta Q(t),$$

(1)

where $Q_0(t) = Q_0 \exp(-t/\tau)$ is the charge on the capacitor in a noiseless RC circuit. So long as $Q_0(t)$ can be distinguished from the fluctuation term $\delta Q(t)$, the discharge process can be followed. When this is no longer possible, it may be said that the capacitor is completely discharged. The problem, then, is to determine the time $t_d$ for which

$$Q_0(t_d) = \langle (\delta Q)^2 \rangle^{1/2},$$

(2)

where $t_d$ is identified as a characteristic total discharge time.

In Sec. II we review three derivations for the well-known result

$$\langle (\delta Q)^2 \rangle = kTC.$$  

(3)

The second derivation explicitly demonstrates that $\delta Q(t)$ may be determined using the noise current $i_n(t)$ as its source. After obtaining the desired expression for $t_d$ in terms of a system $Q$-factor, we discuss in Sec. III several examples of potential interest in the classroom.

II. rms CHARGE FLUCTUATION

A. Linear circuit models

The approach most accessible to introductory students is based on linear circuit models. The usual starting point is to model the noisy resistor in terms of a noiseless resistance of magnitude $R$ either in series with a thermal voltage noise source $v_n(t)$ (Thévenin equivalent circuit), or in parallel with a thermal current noise source $i_n(t)$ (Norton equivalent circuit). Suppose $Q(t)$ is the charge on the capacitor $C$ and $V(t) = Q(t)/C$ is the voltage drop across it. Applying Kirchhoff's voltage law we have for the Thévenin equivalent circuit [Fig. 1(a)],

$$\tau \frac{dV(t)}{dt} + V(t) = v_n(t).$$

(4)

Similarly, for the Norton equivalent circuit [Fig. 1(b)] we have

$$\frac{dQ(t)}{dt} + \frac{Q(t)}{\tau} = i_n(t).$$

(5)

It is explicitly assumed at this point that any fluctuations in the resistor are not due to anything external to the resistor.

Because these equations have the same form, we solve only Eq. (4). The complete solution is of the form

$$V(t) = V_0 \exp(-t/\tau) + v_n(t),$$

(6)

Fig. 1. (a) Thévenin equivalent circuit.
(b) Norton equivalent circuit.
with \( V(0) = V_0 \). To find the particular solution to the inhomogeneous equation, set

\[
v_n(t) = \int_0^\infty A_n(\omega) \exp(i\omega t) d\omega,
\]

(7)

\[
v_p(t) = \int_0^\infty A_p(\omega) \exp(i\omega t) d\omega.
\]

(8)

Then, from Eq. (4), \( A_n \) and \( A_p \) are related through

\[
(i\omega \tau + 1) A_p = A_n
\]

(9)

or

\[
A_p(\omega)/A_n(\omega) = 1/(i\omega \tau + 1) \equiv H(i\omega).
\]

The term \( H(i\omega) \) is known as the system function.

Suppose all we know about \( v_n(t) \) is that

\[
\langle v_n \rangle_t = 0
\]

(10)

and

\[
\langle v_n^2 \rangle_t = \int_0^\infty S_n(\omega) d\omega,
\]

(11)

where \( S_n(\omega) \) is the spectral density function. Then, using straightforward techniques of Fourier analysis, it can be shown that

\[
\langle v_p \rangle_t = 0
\]

(12)

and

\[
\langle v_p^2 \rangle_t = \int_0^\infty |H(i\omega)|^2 S_n(\omega) d\omega.
\]

(13)

For thermal noise we have\(^5\)

\[
S_n(\omega) = 2kTR/\pi,
\]

(14)

so that

\[
\langle v_p^2 \rangle_t = \frac{2kTR}{\pi} \int_0^\infty \frac{d\omega}{1 + (\omega \tau)^2} = kT C.
\]

(15)

Using the relations \( Q = CV \) and Eq. (1) yields

\[
\langle (\delta Q)^2 \rangle_t = C^2 \langle v_p^2 \rangle_t = kTC.
\]

\[\text{B. Microscopic fluctuation model} \]

In Sec. II A the noise source was inserted into an otherwise noiseless circuit. It needs to be demonstrated that the fluctuation term \( \delta Q(t) \) does not itself contribute to \( v_n(t) \) or \( i_n(t) \), because a closer look at the discharge process shows that the fluctuations are not solely governed by the resistor. For example, suppose that the noise current in the resistor is such as to draw too many charges from the capacitor. The voltage across the capacitor will then decrease, reducing the subsequent current. This response produces an apparent dependence between the noise source terms \( v_n(t) \) or \( i_n(t) \) and the fluctuation term \( \delta Q(t) \) of the capacitor. In this section the discharge process is examined in terms of the statistical behavior of individual charges. This leads to an equation for the fluctuation term \( \delta Q(t) \) with \( i_n(t) \) as its source. Subsequent evaluation of \( \langle (\delta Q)^2 \rangle_t \) yields a result identical to that of Sec. II A.

To show that \( i_n(t) \) (or \( v_n \), since \( v_n = i_n R \)) is independent of \( \delta Q(t) \), we follow the approach given by Thornber.\(^4\) The discharge of the capacitor is accomplished by the transfer of individual charges to ground. If we let \( t_i \) be the time at which the \( i \)th electron leaves the capacitor, then the discharge may be expressed as

\[
\frac{dQ(t)}{dt} = -q \sum_i \delta(t - t_i),
\]

(16)

where \( q \) is the electronic charge. Since a given \( t_i \) is dependent on all preceding charge transfers, the \( t_i \) are correlated. To simplify the correlation effects, the current is decomposed into a spontaneous part \( s(t) \) and an induced response part \( r(t) \). The spontaneous part is governed by the noise source, while the induced part is controlled by the instantaneous state \( Q(t) \) of the capacitor. This allows us to rewrite Eq. (16) as

\[
\frac{dQ(t)}{dt} = -qr(t) - qs(t),
\]

(17)

where

\[
s(t) = \sum_i \delta(t - t_i) - r(t).
\]

(18)

We interpret the induced term \( r(t) \) as the dynamical rate of charge loss

\[
qr(t) = Q(t)/RC.
\]

(19)

If there is a fluctuation \( \delta Q \) in \( Q \), then a fluctuation \( \delta r(t) \) occurs in \( r(t) \) also, which to first order in \( \delta Q(t) \) is

\[
\delta r(t) = \frac{\delta Q(t)}{\tau}.
\]

(20)

If we write \( Q(t) \) as in Eq. (1), then Eq. (17) becomes an equation for the fluctuation \( \delta Q(t) \) in terms of a source term:

\[
\frac{d\delta Q(t)}{dt} = -\frac{\delta Q(t)}{\tau} - qs(t).
\]

(21)

The term \( qs(t) \) may now be identified as the noise current \( i_n(t) \), showing that the fluctuation \( \delta Q(t) \) can be determined with \( i_n(t) \) as its source.

The solution to Eq. (21) is

\[
\delta Q(t) = -\int_0^t dt' i_n(t') \frac{\exp \left(-\frac{(t - t')}{\tau}\right)}{\tau}.
\]

(22)

and the squared term is

\[
[\delta Q(t)]^2 = \int_{-\infty}^t dt' \int_{-\infty}^t dt'' i_n(t') i_n(t'') \exp \left(-\frac{(t - t')}{\tau}\right) \exp \left(-\frac{(t - t'')}{\tau}\right).
\]

(23)

Transforming the integration variables to \( u' = (t' - t)/\tau \) and \( u'' = (t'' - t)/\tau \), Eq. (23) becomes

\[
[\delta Q(t)]^2 = \tau^2 \int_0^\infty du' \int_0^\infty du'' \int_{-\infty}^0 du i_n(u' + t) i_n(u'' + t) \exp(u' + u'').
\]

(24)

The mean-square term is

\[
\langle (\delta Q)^2 \rangle_t = \tau^2 \int_0^\infty du' \int_0^\infty du'' \langle i_n(u' + t) i_n(u'' + t) \rangle \exp(u' + u'').
\]

(25)

For pure thermal noise\(^5\)

\[
\langle i_n(u' + t) i_n(u'' + t) \rangle = (2kT/R) \delta(u' - u'').
\]

(26)

Equation (3) follows easily.

\[\text{C. Thermodynamic argument} \]

One of the more elegant methods for reaching this
same conclusion is to consider the capacitor with one degree of freedom in thermal equilibrium with the resistor. Using the equipartition theorem, the residual rms energy stored in the capacitor becomes

$$\frac{1}{2}(\frac{1}{2}kT) = \frac{1}{2}kT.$$  

Equation (3) follows immediately. It is noted that this argument as well as Eq. (14) are valid at all but the lowest temperatures. Nyquist\(^3\) shows how the quantum effects at cryogenic temperatures may be taken into account. But for most applications, these modifications can be neglected, since at room temperature (300 °K) \(k T/kT \approx 1\) for \(k T/kT \geq 6 \times 10^{12}\) Hz.

The three derivations of Eq. (3) reviewed in this section do not by any means exhaust the possibilities.\(^6\) Which one, if any, is chosen for classroom use will depend on the individual class situation.

III. COMPLETE DISCHARGE TIME

At a time for which the capacitor state function \(Q(t)\) can no longer be distinguished from the system fluctuations, we may say that the capacitor is completely discharged. Solving Eq. (2) for \(t_d\) gives

$$t_d = \frac{\tau}{2} \ln \left( \frac{(1/2)Q^2(t)/C}{(1/2)kT} \right) = \frac{\tau}{2} \ln \left( \frac{(1/2)CV^2}{(1/2)kT} \right).$$  

(28)

The argument of the logarithm term is just the ratio of the initial stored energy to the thermal energy, and is defined as the system \(Q\)-factor. Defining the \(dB\) value \(\beta\) as

$$\beta = 10 \log Q,$$  

(29)

we rewrite Eq. (28) as

$$t_d/\tau = \ln(10)/20 \beta \approx 0.115 \beta \approx \log Q.$$  

(30)

For a \(C = 1.0 \mu F\) capacitor, initially charged to \(V_0 = 5.0\) V, discharging through a \(R = 10 \Omega\) resistor at room temperature (\(T = 300 °K\)), we have \(kT = 4.14 \times 10^{-21}\) J, \(\tau = RC = 0.01\) sec, \(Q = 6.03 \times 10^{15}\), \(\beta = 158\), \(t_d/\tau = 18.2\).

In a typical lecture demonstration of the exponential decay of an RC circuit, a capacitor (\(C = 1.0 \mu F\), \(V_0 = 5.0\) V) might be discharged through a high impedance vacuum-tube voltmeter (\(R = 10 \Omega\)). This would give a more easily observed time constant \(\tau = 10\) sec and \(t_d = 182\) sec, leaving \(Q\), \(\beta\), and \(t_d/\tau\) unchanged (depending as they do only on the capacitor variables).

Measurement of the decay time \(t_d\) with a wideband DC amplifier (bandwidth \(\gg 1/RC\)) with unity noise figure would yield the value given by Eq. (28). In this circumstance, the input rms noise to the amplifier is \(\sqrt{kT/C}\). Real amplifiers, however, have noise figures greater than unity. This results in an effective input rms noise greater than \(\sqrt{kT/C}\). Consequently, a measure of \(t_d\), the time for which the capacitor voltage has decayed to the noise level, would yield a value smaller than predicted by Eq. (28). Figure 2 shows an oscilloscope display for a 0.047-\(\mu F\) capacitor, initially charged to a potential of 10 mV, discharging through the 1 M\(\Omega\) input impedance of a Tektronix 5A22N amplifier. The measured \(t_d\) is 300 msec; the calculated value is 490 msec.

In summary, the arguments presented here show that the complete discharge of a capacitor can be described in terms of a decay time \(t_d\), for which the capacitor voltage has decayed to the noise level. The detailed analysis is inappropriate for introductory physics courses, but the results of the analysis together with the thermodynamic argument can be made plausible to the students. While the actual discharge cannot be measured with any real amplifier, the concept can be demonstrated with an oscilloscope with a low-noise high-gain vertical amplifier.

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\(^4\)Present address: Electrical Engineering Technology Dept., Oklahoma State Univ., Stillwater, OK 74074.

\(^5\)See, for example, Paul J. Tipler, Physics (Worth, New York, 1976), pp. 820ff.

\(^6\)J. B. Johnson, Phys. Rev. 32, 97 (1928).

\(^7\)H. Nyquist, Phys. Rev. 32, 110 (1928). Using \(\omega = 2\pi r\), our Eq. (14) would read \(S_a(v) = 4kT\).


\(^9\)See Ref. 4, Appendix A.

\(^6\)See, for example, R. W. Henry, Am. J. Phys. 41, 1361 (1973), or F. Mathieson, ibid. 45, 1184 (1977).