which they ask that the kinetic energy be evaluated and related to the
Poynting vector and the group velocity.

12 H. G. Booker, Cold Plasma Waves (Martinus Nijhoff, Boston, 1984),
Chap. 3.

13 The difference between the average electrical and magnetic energy densi-
ties can be evaluated using the complex Poynting vector \( S = (E \times H^*)/2 \)
[J. A. Stratton, Electromagnetic Theory (McGraw-Hill, New York,
1941), p. 137]. Stratton states that the divergence of the imaginary part
of \( \text{div} \, S \) (his Eq. (33)) is equal to \( 2\omega \) times this difference, implicitly
assuming the conductivity \( \sigma \) to be real, although this is not a requirement
of the derivation. For the case under consideration here, \( \sigma \) is purely imagi-
ary, as indicated by Eq. (C12), and the rhs of Stratton’s equation be-
comes purely imaginary. Further, for a steady-state wave field, the diver-
gence of the complex Poynting vector is zero. With these adjustments,
Stratton’s equation yields the result given here for the difference between
the average electric and magnetic energy densities.

14 G. Schmidt, Physics of High Temperature Plasmas (Academic, New

The random walk method for dc circuit analysis

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A novel method for circuit analysis is presented, which can help in the development of intuition
about the current distribution in a complicated circuit with an emf source and resistors, and can be
used for analytic and numerical calculations.

I. INTRODUCTION

The connection between random walks and potential
ty has been known for some time.1,2 Application to
electric circuits and the equivalence of random walks to
Kirchhoff’s laws are discussed briefly in some more recent
mathematical treatises.3,4 A very nice small book that ap-
peared 5 years ago presents in pedagogical detail the identi-
ity between Markov chains and resistor networks.5 These
presentations are made by mathematicians and seem to be
little known in the physics and electrical engineering com-
munity. To my knowledge, the method has not been used to
help physics students understand the behavior of complex
circuits. In this note the ideas will be presented in the sim-
plest terms and it is hoped that physics teachers may find
this interesting and useful in teaching circuit theory.

For beginning students of electromagnetism, the con-
cepts of charge and force are thought to be easy, while field
and potential are more difficult to comprehend. Simple key
points are the repulsion of like charges and charge conser-
vation. The study of dc currents introduces resistors, wires
(zero resistance), and batteries (source of emf), which are
put together into circuits. A circuit consists of a set of
“nodes” with each node connected to other nodes by resis-
tors. There may be several resistors connecting the same
pair of nodes. Any set of points of the network, mutually
connected by wires, together with those wires themselves,
constitutes a single node. The terminals of one or more
batteries will be connected to nodes. The resistance of a
resistor is usually defined by Ohm’s law i.e., by \( R = \Delta V/I \),
where \( \Delta V \) is the electric potential difference between the
ends of the resistor.

II. THE PROBLEM

It is important for students to understand both quantita-
tively and qualitatively the way the current will be distrib-
uted in a circuit. The simplest example worked out in detail
in a first course is the effective resistance of two resistors in
parallel, \( 1/R = 1/R_1 + 1/R_2 \). This equation follows from
the fact that the current will branch through the two resis-
tors in the ratio of their conductances, i.e., \( I_1/I_2 = S_1/S_2 \),
where the conductance \( S = 1/R \).

The usual method to treat such problems is the use of
Kirchhoff’s node and loop laws. The node law, that there
be no net current into a node, follows from charge conser-
vation and the fact that the repulsion of like charges pro-
hibits the accumulation of net charge at any node or resis-
tor, and is easy for students to understand. The loop law is a
little more difficult since the concepts of \( IR \) drop and emf
are required. The law states that the algebraic sum of \( IR \)
drops around any loop must equal the emf gain of the same
loop. The resulting equations are easy to formulate, but the
solution of the resulting coupled linear system is not trans-
parent and the qualitative nature of the solution is not easi-
ly seen short of doing the calculation.

We present here an alternative method for any case in
which a single source of emf is producing current in a cir-
cuit of any complexity. The circuit will have one node
called the “input” node, where the current enters from the
positive terminal of the battery, and one called the “out-
put” node, where the current exits to the negative terminal.
Between these nodes will be the network of resistors. Ex-
tension to circuits with several batteries is easy.

One way to think about such a circuit physically is to
note that the emf of the battery will produce E fields in the
resistors. The fields “push” the charge, producing currents
that will follow these fields, branching at a node more or
less in the same way as the lines of the field branch out. This
gives a nice picture, but is not very helpful to the intuition
since the bending of the E fields in the wires and resistors
near a node is quite complicated. Since the branching ratio
for currents is independent of the emf for the case we con-
sider (linear Ohm's law), it should not be surprising that the problem can be solved with no reference to e.m.f. or fields at all.

III. THE RANDOM WALK METHOD

The method involves a Monte Carlo procedure in which a single charge jumps from node to adjacent node passing through one resistor with each step. At any node the next jump will take the charge through a resistor connected to that node with a probability proportional to the conductance of that resistor. The probability to jump through resistor \( r_i = S_i/(\Sigma S_j) \), where the sum is over all of the resistors at that node. The additional rule is simply that each charge starts at the input node. When it reaches the output node the walk is finished.

In each completed walk the charge may have jumped back and forth through many of the resistors on the way from the input to the output node. The current for that walk is to be calculated for each resistor, as the net (forward minus backward) motion of the charge through that resistor. The true currents in the network result from the average over as many walks as needed for the desired accuracy. Note that there is no force field acting on the charge. Its drift is entirely due to the designation of the input and output nodes. Note also that Kirchhoff's node law is exactly satisfied by this procedure even for a single walk.

IV. PROOF OF THE METHOD

It is easy to see that Kirchhoff's loop law is also satisfied, but only in the limit of infinitely many walks through the network. To show that result, the appropriate definition of potential for the random walk must be determined. We now use a notation that \( i, j = 1, 2, 3, \ldots \), identify the nodes. A resistor is designated by two indices showing the node pair it connects. An additional index \( \beta \) is needed to distinguish among resistors in the case that there is more than one resistor connecting the same node pair. Its conductance is \( S(i, j, \beta) \) and the current through it is \( I(i, j, \beta) \). That current was previously defined as the net number of times the walk takes the charge from node \( i \) to \( j \) through resistor \( \beta \). In this notation the probability to go from node \( i \) to \( j \) through \( \beta \) is

\[
P(i, j, \beta) = S(i, j, \beta)(\Sigma S_i S(i, j, \beta))^{-1}
\]

and

\[
S(i, j, \beta) = S(j, i, \beta).
\]

The potential \( V(i) \) at each node is defined in terms of \( N(i) \), the number of times the charge stops at node \( i \) during the walk. To be precise,

\[
V(i) = N(i)(\Sigma S_i S(i, j, \beta))^{-1},
\]

where the sum is over all the resistors connected to node \( i \). This definition is for one walk. The true potential is obtained by averaging over infinitely many walks.

The proof follows from the observation that the potential \( V(i) \) at each node, as defined above, will be unique in the limit of many walks, so that the sum of potential differences around any closed loop not containing the battery will vanish as required by the loop law. The proof will now be completed by verifying that the currents will satisfy Ohm's law,

\[
I(i, j, \beta) = [V(i) - V(j)]S(i, j, \beta).
\]

The numbers \( N(i) \) can be interpreted as being proportional to the probability that the charge will be found at node \( i \) during a random walk. Then the current \( I(i, j, \beta) \) from \( i \) to \( j \) through \( \beta \) will be given by the probability for a charge to be on node \( i \) times the probability to jump from \( i \) to \( j \) through resistor \( \beta \), minus the probability for a charge to be on node \( j \) times the probability to jump from \( j \) to \( i \) via \( \beta \). Thus we have

\[
I(i, j, \beta) = N(i)P(i, j, \beta) - N(j)P(j, i, \beta)
= V(i)S(i, j, \beta) - V(j)S(j, i, \beta)
= [V(i) - V(j)]S(i, j, \beta)
\]

in agreement with Eq. (4). The second line follows from Eqs. (1) and (3), and the last from Eq. (2). This completes the proof that the currents from the random walk branch in an identical way to those resulting from solving the equations resulting from Kirchhoff's laws.

The \( N(i) \)'s of Eq. (5), obtained from averaging over many random walks, are not actually probabilities, but are proportional to them. The random walks then give the current branching with no need for potentials, but at the same time Eq. (3) gives numbers proportional to the electric potential at each node, with \( V(output \ node) = 0 \).

V. SIMPLE EXAMPLES

The random walk method can be useful both for numerical applications and in some cases as a simple way of obtaining analytic results. Figures 1–3 show circuits of increasing complexity regarding the random walk.

Each random walk through the circuit of Fig. 1 consists of a single step from the input to the output node. The current branching through \( R_1, R_2, \) and \( R_2 \) is given directly by Eq. (1) as \( I_1/I_2 = S_1/S_2 \). If the current \( I \) in \( R \) is equal to the sum \( I_1 + I_2 \) of those in \( R_1 \) and \( R_2 \), \( R \) is the same as the effective resistance of the parallel combination of \( R_1 \) and \( R_2 \).

\[
\frac{1}{R_{\text{effective}}} = \frac{1}{R_1} + \frac{1}{R_2},
\]

the well-known formula for resistors in parallel.

A random walk for Fig. 2 looks more complicated since the charge may jump back and forth several times through \( R \), before arrival at the output node. However, any of those back and forth walks simply produces a common probability factor, the same for a walk completed via the upper \((R_1, R_2)\) or the lower \((R)\) branch. Thus to compare the

![Fig. 1. Three resistors in parallel.](image-url)
The two-step probability for the upper branch is the product of the probabilities for the separate steps through resistors \( R_1 \) and \( R_2 \). Again here, if the upper \( (I_{12}) \) and lower \( (I) \) branch currents are equal, \( R \) is the same as the effective resistance of the series combination of \( R_1 \) and \( R_2 \). Thus

\[
S = \frac{S_1 S_2}{S_1 + S_2}
\]

or

\[
R_{\text{effective}} = R_1 + R_2,
\]

the well-known formula for two resistors in series.

All networks that can be fully decomposed into series and parallel parts can be solved analytically in the usual way with the use of Eqs. (6) and (9). No numerical procedure is needed, but the random walk method may aid the intuition in finding the large current (high conductance) and small current paths.

**VI. WHEATSTONE BRIDGE**

More complex circuits such as the Wheatstone bridge of Fig. 3 cannot be fully decomposed into parallel and series parts and thus cannot be solved with Eqs. (6)–(9). The conventional treatment requires setting up and solving the coupled linear equations of Kirchhoff's laws. A Monte Carlo implementation of the random walk method is a simpler, more transparent program and is discussed at the end of this section.

However, we first note that the random walk method is useful for obtaining a simple, quick estimate of the current branching. In fact, the method gives directly an analytic, exact solution even for the Wheatstone bridge, so that the Monte Carlo numerical method is really only needed for yet more complicated circuits.

How is this possible? At first glance the random walk looks complicated since there will be both simple paths (in which any node is visited no more than one time), and in addition there will be many complex paths involving jumps back and forth over resistors \( R_1 \), \( R_2 \), and \( R_3 \) including walks around and around the left-hand loop. The Monte Carlo calculation will include all these paths. However, for a Wheatstone bridge circuit, any and all of these complex back and forth or looping additions are common to each of the simple two- and three-step paths through the network and thus contribute a common probability factor that is irrelevant for calculation of ratios of the currents.

For any circuit in which all the complex paths (back and forth and looping) are common to all the simple paths as defined above, the current branching may be calculated exactly, with the use only of the probabilities for the simple paths. This is the case for any Wheatstone bridge circuit and will be demonstrated for the circuit of Fig. 3 below. In general, for a more complicated circuit, the complex paths are not common to all paths, and the analytic method using only the simple paths will not give the correct answer.

For Fig. 3, we label the simple walks by the resistors in the path. The two-step walks are \( W(1,4) \) and \( W(2,5) \) and the three-step walks are \( W(1,3,5) \) and \( W(2,3,4) \). We will focus on calculation of the ratio \( I_1/I_2 \) of the current over resistor \( R_1 = 1 \Omega \) and \( R_2 = 5 \Omega \). Looking only at the first step suggests the estimate \( I_1/I_2 = 5 \). This is, of course, a poor estimate since the charge at the top node cannot get to the right-hand output node as easily as can a charge located at the bottom node. A better estimate is obtained by comparing the probabilities of the most probable simple paths, \( W(1,3,5) \) and \( (2,5) \), which yields

\[
\frac{I_1}{I_2} = \frac{\text{Prob}(1,3,5)}{\text{Prob}(2,5)} = \frac{\frac{5 \times 1 \times 1}{5 \times 1}}{\frac{1 \times 1 \times 1}{5 \times 1}} = \frac{25}{11} = 2.27.
\]

The exact result is obtained from all four of the simple paths as

\[
\frac{I_1}{I_2} = \frac{\text{Prob}(1,3,5) + \text{Prob}(1,5)}{\text{Prob}(2,5) + \text{Prob}(2,3,4)} = 3.0 \quad \text{(Exact)}.
\]

This calculation is clearly much easier than determining the number of coupled equations needed for Kirchhoff's laws, setting them up, and solving them.

For circuits more complicated than that of the Wheatstone bridge, the back and forth paths may not be common to all paths so that a numerical calculation is required. The simplicity of the algorithm needed for a random walk Monte Carlo calculation is shown by the fact that a basic program specifically for the Wheatstone bridge circuit of Fig. 3 requires less than a page of code. The random walk portion contains only 12 lines.

Averaging over 100 walks gives the exact answer of 3.0 to about 10% accuracy and 10,000 walks are needed for 1% precision. The 1% result requires only about a minute on a personal computer or about a second of VAX time. A
VII. SUMMARY AND CONCLUSIONS

By using the random walk method as a way to think about dc circuits, only the simplest concepts of electromagnetism are needed. It is based entirely on the idea that charge cannot collect anywhere in the circuit so that each charge appearing at the input must exit at the output node. Resistance is defined in the most obvious way in terms of jumping probability, and potentials and fields are not needed.

For many circuits analytical results can easily be obtained by this method. Implementation of the Monte Carlo numerical procedure for complex circuits is simple and involves a computer program of only a few lines. If only modest accuracy is required the numerical method can be useful, but if high precision is needed the standard relaxation or linear solver methods will be more efficient.

Finally, this method gives a natural and physical picture of the conduction process. Charges do actually move about all the time, largely in a random fashion, with only a small additional effect of the battery producing the net current. This picture should enable a student to look at a complex circuit, note which are the low-\(R\), high-\(S\) elements, and find the high-probability paths where most of the current will be. It can also aid in thinking about fundamentals even for extremely complex circuits.\(^1-6\)


An inexpensive apparatus for the measurement of the group velocity of light in transparent media using a modified Helium–Neon laser

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An apparatus is described for the measurement of the group velocity of light in transparent media. The apparatus uses a modified He–Ne laser as a stable, amplitude-modulated light source. With the aid of a single beam oscilloscope, results for the velocity of light in air accurate to ~0.02% have been obtained. This accuracy is sufficient to enable the relationship between the group and phase velocities in transparent dispersive media to be experimentally verified.

I. INTRODUCTION

Since 20 October 1983, the meter has been defined as the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second.\(^1\) This definition for the meter fixes the speed of light to be exactly 299 792 458 m/s. However, scientists and students have been attempting to measure the speed of light for more than 300 years and it is doubtful whether laboratory experiments to standardize the meter will hold the same fascination. We therefore present in this article a technique for measuring the velocity of light in air and transparent media in terms of a laboratory length standard and not a technique to standardize the laboratory meter. Using the apparatus described, which can be constructed in any college electronics laboratory, results accurate to better than 0.1% can readily be obtained. The cost of the equipment specific to this experiment is considerably less than the cost of the helium–neon laser.

With the development of cheap helium–neon lasers, the measurement of the velocity of light in schools and universities, using long light paths, has become a realistic proposition. The methods have ranged from electrooptic modulation of the laser beam\(^2-4\) to modified Foucault rotating mirror methods.\(^3\) With path lengths ranging from 1 to 100 m, these methods allow the velocity of light in air to be determined with a typical accuracy of ~1%. Solid-state light-emitting diodes have also been used with pulse modulation\(^5\) and with continuous wave modulation\(^6\) to obtain the velocity of light in air and transparent solids and liquids, again to accuracies ~1%.

The method we present here is one to two orders of magnitude more accurate than these previous methods and relies on the beat frequency of the two internal laser modes to produce a light source amplitude modulated at ultrahigh frequencies. This is a development of a technique first used by Brickner et al.\(^7\) However, whereas Brickner et al. used the direct measurement of the intermediate separation frequency, together with the measured length of the laser cavity to determine the velocity of light, we use the laser simply...