Paradoxical behavior of an infinite ladder network of inductors and capacitors

S. J. van Enk
Norman Bridge Laboratory of Physics, California Institute of Technology 12-33, Pasadena, California 91125

(Received 23 November 1999; accepted 14 February 2000)

The behavior of the impedance of a standard ladder network of capacitors and inductors is analyzed as a function of the size of the network. This behavior may be unstable in the absence of dissipation so that the limit of an infinite network is not well defined. Standard textbooks do not always treat this case correctly.

Paradoxical behavior of an infinite ladder network of inductors and capacitors

There is an elegant way to calculate the input resistance $R$ of an infinite ladder network of resistors of the type shown in Fig. 1 (see, for instance, Ref. 1). Since adding one more step to the beginning of such a network does not change the input resistance—it is after all still the same infinite network—$R$ must equal the resistance of a circuit consisting of $R_1$ placed in series with a parallel combination of $R$ and $R_2$. That is,

$$R = R_1 + \frac{R R_2}{R + R_2},$$

which is easily solved to give

$$R = \frac{1}{2} (R_1 + \sqrt{R_1^2 + 4 R_1 R_2}).$$

Using the same arguments, we can find the complex impedance $Z$ of the infinite ac ladder network consisting of inductors and capacitors shown in Fig. 2. We simply substitute in (2) $i \omega L$ for $R_1$, $1/(i \omega C)$ for $R_2$, and $Z$ for $R$, with the result

$$Z = \frac{1}{2} (i \omega L + \sqrt{-\omega^2 L^2 + 4 L/C}).$$

Apart from the fact that the impedances involved are purely imaginary, there may appear to be nothing new here. But that is not true: Below a certain cutoff frequency, i.e., for $\omega < \omega_c = 2 \sqrt{L/C}$, we see that the complex impedance $Z$ of Eq. (3) has a nonzero real part. Now where did that come from? A circuit consisting solely of components with purely imaginary impedances should have a purely imaginary total impedance $Z$. The real part of $Z$ is, or at least should come as, a surprise.

As it turns out, however, the result (3) is in fact in a subtle way incorrect. In order to see what precisely went wrong on the way to that result, let us consider a finite network of the form of Fig. 2, but terminated after $N$ steps, where each step contains one inductor and one capacitor. Denote the input impedance of such a network by $Z_N$. For definiteness and entertainment we first consider the special case of $\omega = \omega_0 = \sqrt{1/LC}$. As can easily be verified, for $N=1$ we get $Z_1 = i \omega L + 1/(i \omega C) = 0$, for $N=2$ we find $Z_2 = i \omega L$, and $Z_3$ diverges. The fact that $Z_3 \to \infty$ now implies that adding one more step simply gives us back $Z_1$, i.e., $Z_4 = Z_1$. And if we continue, we find for arbitrary integer $M$ that $Z_{3M+N} = Z_N$ for $N=1,2,3$. In other words, the impedance of the network does not converge to a certain limit value, but, in the special case we considered, sequences among three different values. This shows that the seemingly innocuous argument that “adding one more step to the network does not change the impedance” is the culprit. Equation (3), derived from that incorrect argument, cannot be correct either.

On the other hand, the recursive relations

$$Z_{N+1} = i \omega L + \frac{Z_N/(i \omega C)}{Z_N+1/(i \omega C)} = F_\omega(Z_N)$$

with initial condition

$$Z_1 = i \omega L + 1/(i \omega C)$$

do correctly describe networks like the one in Fig. 2 of any finite size $N$ for any frequency $\omega$. As one might anticipate now, for no value of $\omega < \omega_c$ does $Z_N$, as defined by Eqs. (4) and (5), converge in the limit $N \to \infty$. Figure 3 illustrates this for an arbitrarily chosen value of $\omega < \omega_c$. In more technical terms, although the function $F_\omega$ always has one “fixed point” $Z$ with non-negative real part, for $\omega < \omega_c$ this fixed point becomes unstable: $\omega = \omega_c$ is a bifurcation point. This unstable behavior is illustrated in Fig. 4, where the initial condition is chosen close to $Z$, namely $Z_1 = 0.95 Z$, yet the network’s impedance does not converge in the limit $N \to \infty$. Note that choosing an initial value $Z_0$ different from (5) physically corresponds to appending to the end of the finite network an element with an impedance $Z_1 = Z_0$.

Fig. 1. An infinite ladder network consisting of a concatenation of identical segments containing two resistors $R_1$ and $R_2$.

Fig. 2. An infinite ladder network consisting of a concatenation of identical segments containing an inductor with inductance $L$ and a capacitor with capacitance $C$. 

© 2000 American Association of Physics Teachers.
Figure 3 might give the impression that the mapping (4) is chaotic. In addition, the simple fact that for \( \omega = \omega_0 \) the mapping (4) has a periodic point with period three seems to confirm this suspicion. However, the mapping (4) is not continuous and the function \( F_\omega \) does not satisfy the conditions of the theorem "period three implies chaos." The presence of the discontinuity at \( Z = i/(\omega C) \) is indeed relevant here, as period three hits that singularity. Moreover, as can easily be verified, the mapping (4) does not even have a periodic point with period 2, which shows that the universal route to chaos through period doubling is not followed here. As an aside, note that chaos in electric circuits usually refers to the chaotic time evolution of circuits containing nonlinear elements, such as Chua’s circuit.

In practice an LC network like the one of Fig. 2 is used as a transmission line, even for frequencies in the "forbidden" region \( \omega < 2/\sqrt{LC} \). Moreover, the resulting total impedance is, now correctly, given by (3). How can Eq. (3) give the right answer in practice when we just argued that that equation is incorrect? The difference is, of course, that a
realistic inductor has an internal resistance \( r \neq 0 \). What is surprising perhaps is that no matter how small \( r \) is, the impedance \( Z_N \) of the network now does always converge in the limit \( N \to \infty \) to a certain value, as long as \( r \) is nonzero: it just may take many steps to converge. In particular, for negligible \( r \) the limit reached is indeed given by \( \sqrt{6 \ell C} \). This shows that the two limits \( r \to 0 \) and \( N \to \infty \) do not commute.

In this context it is interesting to note that Feynman in Chapter 22 of Ref. 1 discusses the very circuit of Fig. 2 for \( r = 0 \), arrives at Eq. (3) and then, surprisingly, gives an argument why \( Z \) should have a real part. Here is a quote:

But how can the circuit continuously absorb energy, as a resistance does, if it is made only of inductances and capacitances? Answer: Because there is an infinite number of inductances and capacitances, so that when a source is connected to the circuit, it supplies energy to the first inductance and capacitance, then to the second, to the third, and so on. In a circuit of this kind, energy is continually absorbed from the generator at a constant rate and flows constantly out into the network, supplying energy which is stored in the inductances and capacitances down the line.

That explanation is, as we have seen, strictly speaking incorrect, although it does apply to any real-life implementation of the network with a nonzero resistance \( r \).

Finally, Ref. 5 provides another perspective on the same network. Below the cutoff frequency \( \omega_c \), a finite network of size \( N \) has \( N \) resonance frequencies \( \omega_{n(N)} < \omega_c \) for \( n = 1 \cdots N \). That is, the network can sustain oscillations and waves at those frequencies \( \omega_{n(N)} \) without being driven by an external force (voltage). The impedance at such a resonance frequency, therefore, must vanish, i.e., \( Z_N(\omega_{n(N)}) = 0 \). Adding one more step to such a network then gives an impedance of \( Z_{N+1}(\omega_{n(N)}) = i \omega_{n(N)} L \) (since only the first step contributes). Thus we see in a different way that even in the limit of a large network, adding one extra step will change the impedance by a non-negligible amount. Moreover, in the limit \( N \to \infty \) the number of resonances becomes infinite, which once more shows that the result (3) cannot be correct for \( \omega < \omega_c \) in the absence of dissipation.

I thank Jeff Kimble, Ron Legere, and Peter Schlagheck for useful discussions, and the anonymous referees for their constructive comments. This work was funded by the California Institute of Technology and by DARPA through the QUIC (Quantum Information and Computing) program administered by the US Army Research Office, the National Science Foundation, and the Office of Naval Research.