A Re-Examination of the Fundamental Limits on the Radiation Q of Electrically Small Antennas

James S. McLean, *Member, IEEE*

Abstract— **An exact method, which is more straightforward than those previously published, is derived for the calculation of the minimum radiation Q of a general antenna. This expression agrees with the previously published and widely cited approximate expression in the extreme lower limit of electrical size. However, for the upper end of the range of electrical size which is considered electrically small, the exact expression given here is significantly different from the approximate expression. This result has implications on both the bandwidth and efficiency limitations of antennas which fall into this category.**

I. Introduction

Electrically small antennas are antennas with geometrical dimensions which are small compared to the wavelengths of the electromagnetic fields they radiate. More specifically, the term "electrically small antenna" has become understood to include any antenna which fits inside a sphere of radius $a = 1/k$ where k is the wave number associated with the electromagnetic field [1], [2]. The radiative properties of such electrically small antennas were first investigated by Wheeler [3] who coined the term "radiation power factor." Later, a very comprehensive theory was presented by Chu [4] in which the minimum radiation quality factor Q of an antenna, which fits inside a sphere of a given radius, was derived. This approximate theory was later extended by Harrington [5] to include circularly polarized antennas. Collin [6] and later, Fante [7], published an exact theory based on a calculation of the evanescent energy stored around an antenna. More recently, a comprehensive review paper was published by Hansen [2]. The approximate fundamental lower limit given by Chu and Harrington's theory for the radiation Q of an antenna of a given size has become, by far, the most widely cited; it has been included in a review article [2], at least one antenna theory textbook [8], and a popular antenna design handbook [9].

It is useful to re-examine this fundamental limit for two reasons. First, in that it is a fundamental limit, it should be computed to the highest possible accuracy; it makes little sense to speak of approximate fundamental limits. Second, from a pedagogical viewpoint, understanding of the concept of the minimum radiation Q of an antenna may be enhanced by the alternative derivation given here. In this brief communication, we will first review the previously published approximate theory. Then, we will calculate the radiation Q directly from the fields of the TM_{01} spherical mode (or equivalently, those of a short, linear, current element) using a new, more direct technique and show that the result is exactly the same as that obtained using either an equivalent ladder-network analysis with no approximations or the exact field-based technique given by Collin and Fante. Finally, we will compare the new approach with that given by Collin and Fante.

Manuscript received April 26, 1994; revised October 20, 1995.

The author is with the Department of Electrical and Computer Engineering, University of Wisconsin, Madison, WI 53706 USA.

Publisher Item Identifier S 0018-926X(96)03225-5.

II. Summary of Approximate Theory

The radiation Q of a simple antenna is not obviously defined since, in general, such an antenna is not self-resonant. Strictly speaking, the Q of a system is defined to be 2π times the ratio of the maximum energy stored to the total energy lost per period. For an antenna, the following definition for radiation Q is generally accepted [4], [5]

$$Q = \begin{cases} \dfrac{2\omega W_e}{P_{\mathrm{rad}}} & W_e > W_m \\ \dfrac{2\omega W_m}{P_{\mathrm{rad}}} & W_m > W_e. \end{cases} \tag{1}$$

where W_e is the time-average, nonpropagating, stored electric energy, W_m is the time-average, nonpropagating, stored magnetic energy, ω denotes radian frequency, and P_{rad} denotes radiated power. The basis for this definition is that it is implicitly assumed that the antenna will be resonated with an appropriate lossless circuit element to effect a purely real input impedance at a specific design frequency. Thus, the definition of the radiation Q of an antenna is similar to the definition of Q for a practical circuit element, which stores predominantly one form of energy while exhibiting some losses.

In Chu's theory, the antenna is enclosed by a sphere of radius a, the smallest possible sphere which completely encloses the antenna. The fields of the antenna external to the sphere are represented in terms of a weighted sum of spherical wave functions, the so-called "modes of free space" [10]. It is implicit in Chu's work that these modes exhibit power orthogonality; that is, they carry power independently of one another, just as modes in a uniform metallic waveguide. From the spherical wave-function expansion, the radiation Q is calculated in terms of the time-average, nonpropagating energy *external* to the sphere and the radiated power. In this manner, the radiation Q so calculated will be the minimum possible radiation Q for any antenna which fits in the sphere; any energy stored within the sphere could only increase the Q. However, the calculation of this radiation Q is not straightforward because the *total* time-average stored energy outside the sphere is infinite just as it is for any propagating wave or combination of propagating waves and nonpropagating fields [4]. Some technique for separating the nonpropagating energy from the total energy is required. It is not possible to calculate the nonpropagating stored energy using simply the near-field electric and magnetic-field components because energy is a nonlinear quantity [4]. Instead of working directly with the electric and magnetic fields, Chu derives an equivalent ladder network for each spherical waveguide mode using a clever technique based on the recurrence relations for the spherical Bessel functions and a continued fraction expansion. From the equivalent circuit, it would be possible to calculate the total nonpropagating energy and, hence, the radiation Q, by summing up the electric and magnetic energies stored in the inductances and the capacitances. However, as Chu points out in his paper, this would be quite tedious for all but the lowest order modes. Therefore, Chu derives an equivalent second-order series *RLC* circuit and calculates the Q from this equivalent circuit, assuming it behaves as a *lumped* second-order network over some limited range of frequency. It is worthwhile to note that this is a significant approximation.

From Chu's calculations, it has been shown that an antenna which excites only the $n = 1$ mode (either TE_{01} or TM_{01}) external to the sphere (and, of course, stores no energy in the sphere) has the lowest possible radiation Q of any linearly polarized antenna. An approximate analytical expression for this Q has been given [2] but appears to be incorrect.[1]

III. Exact Derivation of Radiation Q from Nonpropagating Energy

To derive an exact expression for radiation Q, we begin with the fields of the TM_{01} spherical mode with even symmetry about $\theta = 0$. These fields can be obtained from an r-directed magnetic vector potential, A_r [10]. We also note that they are equivalent to the fields of a short, linear electric-current element.

$$A_r = -\cos\theta\, e^{-jkr}\left(1 - \frac{j}{kr}\right) \tag{2}$$

$$H_\phi = \sin\theta\, e^{-jkr}\left(\frac{j}{kr^2} - \frac{1}{r}\right) \tag{3}$$

$$E_\theta = \frac{1}{j\omega\epsilon}\sin\theta\, e^{-jkr}\left(-\frac{1}{r^2} - \frac{jk}{r} + \frac{j}{kr^3}\right) \tag{4}$$

$$E_r = \frac{1}{\omega\epsilon} 2\cos\theta\, e^{-jkr}\left(\frac{1}{kr^3} + \frac{j}{r^2}\right). \tag{5}$$

[1] References [2], [8], and [9] give

$$Q = \frac{1 + 3k^2a^2}{k^3a^3[1 + k^2a^2]}.$$

There appears to be an algebraic error in this expression. Working from Chu's equivalent second-order network and making the approximations indicated in Chu's paper, one obtains

$$Q = \frac{1 + 2k^2a^2}{k^3a^3[1 + k^2a^2]}.$$

Of course, both expressions agree in the lower limit of ka. However, they begin to differ significantly as ka approaches one (the upper limit of what is considered electrically small).

Here, the field components are to be taken as root-mean-square (RMS) values. From these field components we calculate the electric- and magnetic-energy densities w_e and w_m

$$w_e = \frac{1}{2}\epsilon \vec{E}\cdot\vec{E}^* \tag{6}$$

$$= \frac{1}{2}\epsilon(|E_\theta|^2 + |E_r|^2) \tag{7}$$

$$= \frac{1}{\omega}\eta\frac{1}{2}\left[\sin^2\theta\left(\frac{1}{k^3r^6} - \frac{1}{kr^4} + \frac{k}{r^2}\right) + 4\cos^2\theta\left(\frac{1}{k^3r^6} + \frac{1}{kr^4}\right)\right] \quad \text{and} \tag{8}$$

$$w_m = \frac{1}{2}\mu\vec{H}\cdot\vec{H}^* \tag{9}$$

$$= \frac{1}{2}\mu|H_\phi|^2 \tag{10}$$

$$= \frac{1}{2}\mu\sin^2\theta\left(\frac{1}{k^2r^4} + \frac{1}{r^2}\right) \tag{11}$$

where $\eta = \sqrt{\mu/\epsilon}$. Now consider the electric-energy density associated with the traveling wave; that is, the energy calculated from the field components which produce radiated power. This could be called the propagating energy density, w_e^{rad}. This energy density is computed using only the radiation fields

$$H_\phi^{\mathrm{rad}} = -\sin\theta\frac{e^{-jkr}}{r} \tag{12}$$

$$E_\theta^{\mathrm{rad}} = -\eta\sin\theta\frac{e^{-jkr}}{r} \tag{13}$$

$$w_e^{\mathrm{rad}} = \frac{1}{2}\epsilon|E_\theta^{\mathrm{rad}}|^2 = \frac{\eta^2}{r^2}\sin^2\theta. \tag{14}$$

If we define the nonpropagating electric-energy density w_e' as the difference between the total electric-energy density and the propagating electric-energy density we obtain

$$w_e' = w_e - w_e^{\mathrm{rad}} = \frac{\eta}{2\omega}\left[\sin^2\theta\left(\frac{1}{k^3r^6} - \frac{1}{kr^4}\right) + 4\cos^2\theta\left(\frac{1}{k^3r^6} + \frac{1}{kr^4}\right)\right]. \tag{15}$$

The total nonpropagating electric energy W_e' is

$$W_e' = \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty} w_e'\, r^2\sin\theta\, dr\, d\theta\, d\phi \tag{16}$$

$$= \frac{4\pi\eta}{3\omega}\left[\frac{1}{k^3a^3} + \frac{1}{ka}\right]. \tag{17}$$

The total radiated power may be determined by integrating the real part of the Poynting vector over a spherical surface of any radius

$$P_{\mathrm{rad}} = \int_0^{2\pi}\int_0^{\pi} \mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*)\cdot\hat{a}_r\, r^2\sin\theta\, d\theta\, d\phi. \tag{18}$$

When the integration is carried out, the radiated power is found to be

$$P_{\mathrm{rad}} = \frac{8\pi}{3}\eta. \tag{19}$$

The quality factor Q is then

$$Q = \frac{2\omega W_e'}{P_{\mathrm{rad}}} = \frac{1}{k^3a^3} + \frac{1}{ka}. \tag{20}$$

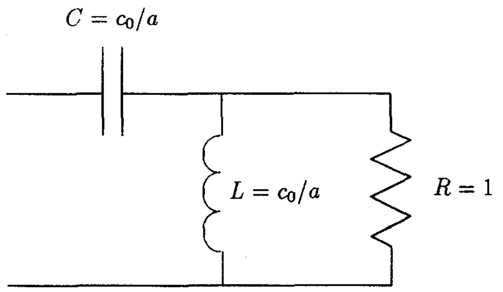

Fig. 1. Equivalent circuit for TM_{01} spherical mode.

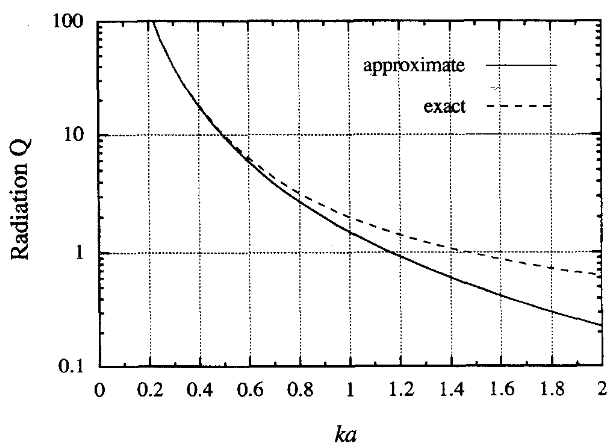

Fig. 2. Radiation Q associated with TE_{01} or TM_{01} spherical mode; minimum attainable radiation Q of a linearly polarized antenna. The approximate expression is that derived in [4] and given in the footnote of this paper; the exact expression is that given in (20).

The quality factor associated with the TM_{01} mode may also be calculated easily from Chu's equivalent ladder network for this mode, which is shown in Fig. 1. The total electric energy stored in the circuit is

$$W_e' = \frac{1}{2}C|V_c|^2 = \frac{1}{2\omega}\frac{1}{ka}. \tag{21}$$

The power dissipated in the resistor is

$$P_r = |I_r|^2 R = \frac{k^2a^2}{1+k^2a^2}. \tag{22}$$

Therefore, the Q is

$$Q = \frac{2\omega W_e}{P_r} = \frac{1}{k^3a^3} + \frac{1}{ka}. \tag{23}$$

This is in perfect agreement with (20), as well as the exact expression given in [6] and [7]. The exact expression for the radiation Q associated with the TM_{01} mode is plotted versus ka in Fig. 2, along with the approximate expression given in [4]. As can be seen from the graph, the two expressions agree for vanishingly small values of ka. However, at $ka = 1$ they differ by a factor of 4/3.

IV. Circularly-Polarized Electrically Small Antennas

TE_{0n} and TM_{0n} fields can be combined (with the appropriate complex weighting) to produce circularly polarized fields. It has been shown that the lowest achievable radiation Q for a circularly polarized antenna is given by that of the combination of TE_{01} and TM_{01} modes, which produces circular polarization [5]. However, the calculation in [5] of the radiation Q of this combination of modes was performed using the same approximation used in [4] for deriving the minimum radiation Q of a linearly polarized antenna. While an exact expression has been given in [6] and [7], it is useful to derive the exact expression here using the new, more direct technique.

Since the TE_{01} is the dual of the TM_{01} mode, the fields of the TE_{01} spherical mode with even symmetry about $\theta = 0$ can be obtained from an r-directed electric vector potential, F_r [5]

$$F_r = -\cos\theta\, e^{-jkr}\left(1 - \frac{j}{kr}\right) \tag{24}$$

$$E_\phi = \sin\theta\, e^{-jkr}\left(\frac{j}{kr^2} - \frac{1}{r}\right) \tag{25}$$

$$H_\theta = \frac{1}{j\omega\mu}\sin\theta\, e^{-jkr}\left(-\frac{1}{r^2} - \frac{jk}{r} + \frac{j}{kr^3}\right) \tag{26}$$

$$H_r = \frac{1}{\omega\mu}2\cos\theta\, e^{-jkr}\left(\frac{1}{kr^3} + \frac{j}{r^2}\right). \tag{27}$$

If we combine TE_{01} and TM_{01} fields of the proper amplitude and phase we can obtain circular polarization in the far-field region. This would require a complex amplitude of $j\eta$ for E_ϕ. We can calculate the radiation Q using the same definition given earlier. In this case, the total electric energy density is

$$w_e = \tfrac{1}{2}\epsilon(|E_\theta|^2 + |E_r|^2 + |E_\phi|^2). \tag{28}$$

The total nonpropagating electric energy is then

$$W_e' = \frac{4\pi\eta}{3\omega}\left[\frac{1}{k^3a^3} + \frac{2}{ka}\right]. \tag{29}$$

The radiated power for each mode is equal and, therefore, the total radiated power is twice that of the TM_{01} mode acting alone. Therefore, the radiation Q is

$$Q = \frac{1}{2}\left(\frac{1}{k^3a^3} + \frac{2}{ka}\right). \tag{30}$$

Again, this expression for the radiation Q is in perfect agreement with that which can be obtained from the exact equivalent circuits of the TM_{01} and TE_{01} spherical modes. For very small values of ka this is approximately one-half of the radiation Q associated with the the TM_{01} mode acting alone. If the Q is calculated from an approximate equivalent series RLC network for the two modes, the following approximate expression for Q is obtained

$$Q = \frac{1}{2}\left[\frac{1+3k^2a^2}{k^3a^3(1+k^2a^2)}\right]. \tag{31}$$

This is the expression plotted in [5] for the minimum radiation Q of a circularly polarized antenna. It is plotted in Fig. 3 along with the exact expression given here. As in the case of the linearly polarized antenna, the two expressions agree in the lower limit of ka but differ significantly from each other for values of ka near one. Finally, we note that the minimum radiation Q of a circularly polarized antenna is only approximately half that of a linearly polarized antenna. This is because the TE_{01}

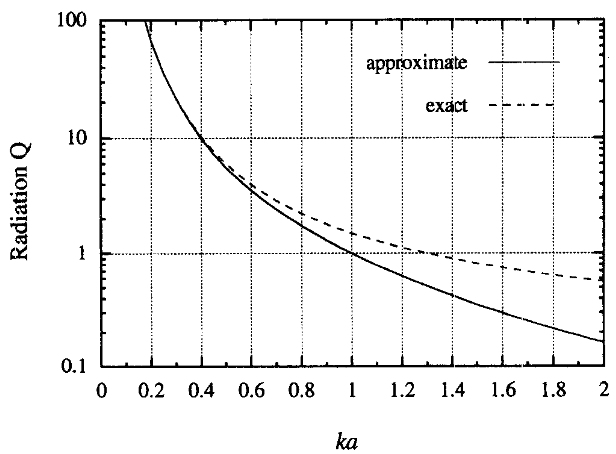

Fig. 3. Radiation Q associated with circularly polarized combination of TE_{01} and TM_{01} modes; minimum attainable radiation Q for a circularly polarized antenna. The approximate expression is that given in (31); the exact expression is that given in (30).

mode, while storing predominantly magnetic energy in the nonradiating fields, also stores some electric energy. Likewise the TM_{01} mode stores predominantly electric energy but also stores some magnetic energy in its nonradiating fields.

V. Comparison with Other Exact Methods

As noted earlier, similar results for the radiation Q of the TM_{01} and TE_{01} modes have been derived and published by Collin [6] and Fante [7]. However, the method they used to compute the nonpropagating energies differs from that used here. Succinctly, Collin and Fante first calculated an energy density associated with the radiating fields $w_e^{\rm rad} + w_m^{\rm rad}$ by taking the ratio of the power flow and the velocity of energy propagation which they took to be c_0. Next, they computed the sum of the nonpropagating electric and magnetic energies $W_e' + W_m'$ by extracting the energy density associated with the radiating field from the total energy density and then integrating the result. They then calculated the difference between the nonpropagating electric and magnetic energies using Poynting's theorem. From these two quantities, the nonpropagating electric and magnetic energies may be determined. In this paper, it has been shown that the electric and magnetic energies can be computed independently of one another. The author feels that this technique is not in disagreement with the fundamental relations upon which the theory of Collin and Fante is based. As was noted by Collin, the electric and magnetic energy densities of the radiating fields must be equal; that is, equipartition of energy must hold for the radiating fields. Therefore, the total energy density of the radiating fields must simply be twice either the electric- or magnetic-energy density associated with the radiating fields. Since the total electric- and magnetic-energy densities are known and the electric and magnetic energies associated with the radiating fields are known, it must be possible to compute the nonpropagating electric- and magnetic-energy densities independently of one another.

VI. Conclusion

An exact calculation of the radiation Q associated with the TM_{01} and TE_{01} spherical modes, and hence, the minimum attainable radiation Q of a linearly polarized antenna has been given. This exact expression differs somewhat from the previously given approximate expression. For vanishingly small values of ka, this radiation Q is approximately given by

$$Q \approx \frac{1}{k^3a^3} \tag{32}$$

and thus is in agreement with Chu's [4] theory and Wheeler's [3] theory for this range of ka. However, for values of ka near one (but still less than one), the expressions differ significantly. This range of electrical size is important in that a number of antenna designs, such as multiply folded linear antennas, fall into this range. For example, the largest electrical dimension of Goubau's [11] folded, loaded monopole is about $2/k$ at the center frequency of the antenna's operating range; it fits snugly into a radiansphere at this frequency. From the exact expression given here, it would appear that the lowest achievable radiation Q of an antenna of this electrical size is actually somewhat larger than previously thought. As has been previously noted [2], [4], the relation between the radiation Q and the maximum achievable bandwidth is not straightforward. Nonetheless, a larger radiation Q would still imply a narrower maximum achievable bandwidth. Therefore, the restriction on bandwidth for an electrically small linearly polarized antenna is actually somewhat more restrictive than is implied by Chu's approximate theory. Finally, we note that the exact expression for minimum antenna Q can be derived three different ways yielding identical results.

Acknowledgment

The author would like to thank Prof. H. Foltz, University of Texas–Pan American and Prof. G. Crook, University of Wisconsin–Madison, for their help, as well as the constructive comments of the anonymous reviewers.

References

[1] H. A. Wheeler, "Small antennas," *IEEE Trans. Antennas Propagat.*, vol. AP-23, pp. 462–469, July 1975.

[2] R. C. Hansen, "Fundamental limitations in antennas," *Proc. IEEE*, vol. 69, pp. 170–182, Feb. 1981.

[3] H. A. Wheeler, "Fundmental limitations of small antennas," *Proc. IRE*, vol. 35, pp. 1479–1484, Dec. 1947.

[4] L. J. Chu, "Physical limitations on omni-directional antennas," *J. Appl. Phys.*, vol. 19, pp. 1163–1175, Dec. 1948.

[5] R. F. Harrington, "Effect of antenna size on gain, bandwidth and efficiency," *J. Res. Nat. Bur. Stand.*, vol. 64-D, pp. 1–12, Jan./Feb. 1960.

[6] R. E. Collin and S. Rothschild, "Evaluation of antenna Q," *IEEE Trans. Antennas Propagat.*, vol. AP-12, pp. 23–27, Jan. 1964.

[7] R. L. Fante, "Quality factor of general ideal antennas," *IEEE Trans. Antennas Propagat.*, vol. AP-17, pp. 151–155, Mar. 1969.

[8] C. A. Balanis, *Antenna Theory, Analysis, and Design.* New York: Wiley, 1982, pp. 439–444.

[9] P. K. Park and C. T. Tai, "Receiving antennas," in *Antenna Handbook: Theory, Applications, and Design*, Y. T. Lo and S. W. Lee, Ed. New York: Van Nostrand Reinhold, 1998, pp. 6.22–6.24.

[10] R. F. Harrington, *Time-Harmonic Electromagnetic Fields.* New York: McGraw-Hill, 1961.

[11] G. Goubau, "Multi-element monopole antennas," in *Proc. Workshop Electrically Small Antennas, ECOM*, Ft. Monmouth, NJ, pp. 63–67, May 1976.

James S. McLean (M'90) received the B.S., M.S., and Ph.D. degrees in electrical engineering from the University of Texas, Austin, in 1984, 1986, and 1990, respectively.

He has worked with TRW Antenna Systems Laboratory, Redondo Beach, CA, and is currently an Associate Scientist in the Electrical and Computer Engineering Department at the University of Wisconsin–Madison. His current research interests include electrically small and low-profile antennas for wireless and mobile communications.